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Extensions and solutions for nonlinear diffusion equations and random walks

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We investigate a connection between random walks and nonlinear diffusion equations within the framework proposed by Einstein to explain the Brownian motion. We show here how to properly modify that framework in order to handle different physical scenarios. We obtain solutions for nonlinear diffusion equations that emerge from the random walk approach and analyse possible connections with a generalized thermostatistics formalism. Finally, we conclude that fractal and fractional derivatives may emerge in the context of nonlinear diffusion equations, depending on the choice of distribution functions related to the spreading of systems.

1. Introduction

In 1905, Karl Pearson (1857–1936) proposed an intriguing question that ends up having profound implications in several fields of science. In *Nature* [1], he has asked for help in finding the solution to a problem that is now known as the *random walk* or the *drunkard's walk*. It was Lord Rayleigh (1842–1919) who provided a suitable solution to that problem, and ever since the concept of random walk has permeated several disciplines and its central is Statistical Physics.

Indeed, the idealization of a path realized by successive random steps, namely a *random walk*, has been used as a model for different stochastic processes, particularly to describe diffusive phenomena. This description provides a suitable way of understanding extensions of diffusion equations both in terms of distributions' functions associated with the walkers dynamics, and differences from standard Markovian stochastic processes, which are characterized by a linear time dependence of the mean square displacement, i.e. $\langle(\Delta x)^2\rangle \sim t$.

Within this context, different extensions of random walks have been analysed to obtain generalized diffusion equations, such as the ones with fractional-time or spatial derivatives [2,3]. These fractional equations emerge from random walk approaches when the waiting time or jumping distributions have long-tailed behaviours [4,5]. Fractional diffusion equations have been successfully applied for modelling challenging problems, such as transport through porous media [6,7], dynamic processes in protein folding [8], infiltration [9], single-particle tracking [10], electrical response [11–13], and diffusion on fractals [14].

On the other hand, nonlinear diffusion equations [15–27] can also arise from random walk processes after suitable modifications in the standard approach and have been investigated in connection with the Tsallis statistics [28] where, under appropriate conditions, they can be associated with Lévy distributions [29–31]. Nonlinear equations can also be obtained from other statistical approaches [32,33] and have been applied to a wide variety of problems [34–41]. In addition to diffusive phenomena, interactions among particles may occur and cause, for instance, the consumption or production of particles such as in the case of chemical reactions. Particles can also interact in different forms so that only the process of relaxation of a given part of a system is altered. An example of the former situation appears when part of the particles acts as a thermal bath to the other during the relaxation process [42].

Here, we investigate how to use the random walk approach to obtain nonlinear diffusion equations that take into account the interactions among walkers. We obtain exact solutions for these equations under different conditions, such as in the absence (where we explore the role and relevance of fractional derivatives) or in the presence of external forces with a linear dependence on the spatial variable, and solutions for the nonlinear diffusion equations emerging from the problem of interacting walkers.

The rest of this paper is organized as follows. In §2, we present all analyses and exact results. In §3, we conclude this work with a discussion of our findings and some final remarks.

2. Random walk and nonlinear diffusion equation

We start by asking how to obtain different nonlinear diffusion equations from a *random walk* perceptive. To do so, we briefly review the formulation of Einstein [43] used to explain the Brownian motion, which employs the following equation:

$$\rho(x,t+\tau) = \int_{-\infty}^{\infty} \rho(x-x',t)\Phi(x')\,\mathrm{d}x', \tag{2.1}$$

where $\rho(x,t)$ is the number of particles per unit volume and $\Phi(x')\,\mathrm{d}x'$ is the probability of a particle moving from x' to $x'+\mathrm{d}x'$ during the time interval τ. This probability is normalized and symmetric, i.e. $\int_{-\infty}^{\infty}\Phi(x')\,\mathrm{d}x' = 1$, with $\Phi(x') = \Phi(-x')$. For very small τ, we can approximate $\rho(x,t+\tau)$ in equation (2.1) by $\rho(x,t+\tau) \simeq \rho(x,t) + \tau\partial_t\rho(x,t)$, and $\rho(x-x')$ in the right-hand side of equation (2.1) can be expanded (for a distribution $\Phi(x')$ that decays very fast) to second-order terms in x', yielding $\rho(x-x',t) = \rho(x,t) - x'\partial_x\rho(x,t) + (x'^2/2!)$ $\partial_x^2\rho(x,t) + \cdots$.

By using these approximations in equation (2.1), we thus obtain the diffusion equation

$$\frac{\partial}{\partial t}\rho(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}\rho(x,t), \tag{2.2}$$

where $\mathcal{D} = \langle x'^2\rangle/(2\tau)$ is the diffusion coefficient and $\langle x'^2\rangle = \int_{-\infty}^{\infty} x'^2\Phi(x')\,\mathrm{d}x'$ is second moment of the probability distribution. By following the same approach, we can also obtain different nonlinear diffusion equations. For instance, if we include a nonlinear dependence in ρ into the dispersal term ($\Phi \to \Phi(\rho, x)$) the jumping probability will also exhibit a nonlinear behaviour, leading to memory effect over the diffusive process. In particular, for $\Phi \to \Psi(\rho)\Phi(x')$, we find

$$\rho(x,t+\tau) = \int_{-\infty}^{\infty} \Psi[\rho(x-x',t)]\rho(x-x',t)\Phi(x')\,\mathrm{d}x' + \{1-\Psi[\rho(x,t)]\}\,\rho(x,t). \tag{2.3}$$

If we now take the limit $\tau \to 0$ and $x' \to 0$ in the previous set of equations, we obtain

$$\frac{\partial}{\partial t}\rho(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}\left\{\Psi[\rho(x,t)]\rho(x,t)\right\}, \tag{2.4}$$

that is, a nonlinear diffusion equation that is useful to describe several problems depending on the choice of $\Psi[\rho(x,t)]$. Another interesting possibility occurs when $\Phi(x)$ depends explicitly on $\rho(x,t)$. In this case, the definition of the diffusion coefficient is different from the previous one, as detailed in [44].

It is worth noticing that equation (2.4) is equal to equation (2.2) when $\Psi(\rho) = 1$ and thus yields a usual diffusion process. On the other hand, for $\Psi(\rho) = \rho^{\nu-1}$, we obtain $\partial_t\rho(x,t) = \mathcal{D}\partial_x^2\rho^{\nu}(x,t)$, where the diffusive term presents a nonlinear dependence on the distribution $\rho(x,t)$. This is a characteristic of an anomalous correlated diffusion that can be associated with a generalized thermostatistics formalism [28,45] based on the entropy

$$S_q = -\frac{1}{1-q}\left(1 - \int \mathrm{d}x\rho^q(x,t)\right), \tag{2.5}$$

from which we have $S_q^{A\cup B} = S_q^A + S_q^B + (1-q)S_q^A S_q^B$ and $q \to 1$ recovers the standard entropy results.

Naturally, other choices for $\Psi(\rho)$ and $\Phi(x)$ are possible. In particular, different diffusive behaviours (one characterized by an usual and others by anomalous diffusion) emerge for $\Psi(\rho) = 1 + \alpha\rho^{\nu-1} + \tilde{\alpha}\rho^{\tilde{\nu}-1}$, where the parameters α and $\tilde{\alpha}$ control the influence of the nonlinear terms. This form for $\Psi(\rho)$ has applications in the spatial distribution of dispersing animals [46,47] and in the overdamped motion of interacting particles [48].

We further observe that $\Phi(z)$ has a remarkable influence on the diffusion process, and its choice leads to significantly different behaviours of the respective solution. That is the case of frameworks related to spatial and fractional-time derivatives, which emerge when $\Phi(z)$ is a power-law function. For instance, by considering $\Phi(z) \propto 1/|z|^{1+\mu}$ in the linear case, we find the results of ref. [49], where fractional spatial derivatives are present. Another possibility is to consider spatial or time dependence or both in Ψ. In this case, we can write equation (2.3) as

$$\begin{aligned}\rho(x,t+\tau) = &\int_{-\infty}^{\infty} \Psi[x-x',t;\rho(x-x',t)]\rho(x-x',t)\Phi(x')\,\mathrm{d}x' \\ &+ (1-\Psi[x,t;\rho(x,t)])\,\rho(x,t).\end{aligned} \tag{2.6}$$

In the limits $\tau \to 0$ and $x' \to 0$, equation (2.6) simplifies to

$$\frac{\partial}{\partial t}\rho(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}\left\{\Psi[x,t;\rho(x,t)]\rho(x,t)\right\}, \tag{2.7}$$

and it is related to the Ito–Langevin equation

$$\dot{x}(t) = \sqrt{\mathcal{D}\Psi[x,t;\rho(x,t)]}\,\xi(t), \tag{2.8}$$

where $\xi(t)$ is the Gaussian white noise with $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle \propto \delta(t-t')$. We notice that the trajectory described by equation (2.8) is simultaneously determined by equations (2.7) and (2.8).

We can also extend equation (2.7) using the Ito–Langevin approach to account for the presence of external forces by incorporating the term $F[x,t;\rho(x,t)]$ into equation (2.8), thus yielding

$$\dot{x}(t) = F[x,t;\rho(x,t)] + \sqrt{\mathcal{D}\Psi[x,t;\rho(x,t)]}\xi(t), \tag{2.9}$$

and consequently,

$$\frac{\partial}{\partial t}\rho(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}\{\Psi[x,t;\rho(x,t)]\rho(x,t)\} - \frac{\partial}{\partial x}\{F[x,t;\rho(x,t)]\rho(x,t)\}. \tag{2.10}$$

We can also obtain the previous equation from equation (2.6) by writing

$$\begin{aligned}\rho(x,t+\tau) = &\int_{-\infty}^{\infty}\Psi_{-}[x-x',t;\rho(x-x',t)]\rho(x-x',t)\Phi_{-}(x')\,\mathrm{d}x' + (1-\Psi_t[x,t;\rho(x,t)])\,\rho(x,t)\\ &+\int_{-\infty}^{\infty}\Psi_{+}[x+x',t;\rho(x+x',t)]\rho(x+x',t)\Phi_{+}(x')\,\mathrm{d}x',\end{aligned} \tag{2.11}$$

where $\Psi_t[x,t;\rho] = \Psi_{-}[x,t;\rho] + \Psi_{+}[x,t;\rho]$. In equation (2.11), we assume that the probability distributions $\Phi_{-}(x)$ and $\Phi_{+}(x)$ are normalized, i.e. $\int_{-\infty}^{\infty}\mathrm{d}x'\Phi_{-}(x') = \int_{-\infty}^{\infty}\mathrm{d}x'\Phi_{+}(x') = 1$, and we relax the condition of symmetry previously imposed to derive equation (2.2). By considering the limit $\tau\to 0$ and $x'\to 0$ and performing an expansion, we find from equation (2.11)

$$\begin{aligned}\frac{\partial}{\partial t}\rho(x,t) = &-\frac{\partial}{\partial x}\{\mathcal{F}_{-}\Psi_{-}[x,t;\rho(x,t)] - \mathcal{F}_{+}\Psi_{+}[x,t;\rho(x,t)]\}\,\rho(x,t)\\ &+\frac{\partial^2}{\partial x^2}\{\mathcal{D}_{-}\Psi_{-}[x,t;\rho(x,t)] + \mathcal{D}_{+}\Psi_{+}[x,t;\rho(x,t)]\}\,\rho(x,t),\end{aligned} \tag{2.12}$$

where

$$\mathcal{F}_{-(+)} = \frac{1}{\tau}\langle x'\rangle_{-(+)} \quad \text{with } \langle x'\rangle_{-(+)} = \int_{-\infty}^{\infty} x'\Phi_{-(+)}(x')\,\mathrm{d}x'$$

and

$$\mathcal{D}_{-(+)} = \frac{1}{2\tau}\langle x'^2\rangle_{-(+)} \quad \text{with } \langle x'^2\rangle_{-(+)} = \int_{-\infty}^{\infty} x'^2\Phi_{-(+)}(x')\,\mathrm{d}x'.$$

By choosing

$$\Psi_{-}[x,t;\rho(x,t)] = \frac{(\mathcal{F}_{+}\Psi[x,t;\rho(x,t)] + \mathcal{D}_{+}F[x,t;\rho(x,t)])}{\Delta} \tag{2.13}$$

and

$$\Psi_{+}[x,t;\rho(x,t)] = \frac{(\mathcal{F}_{-}\Psi[x,t;\rho(x,t)] - \mathcal{D}_{-}F[x,t;\rho(x,t)])}{\Delta} \tag{2.14}$$

with $\Delta = \mathcal{F}_{+}\mathcal{D}_{-} + \mathcal{D}_{+}\mathcal{F}_{-}$, equation (2.12), we formally recover equation (2.10).

Equation (2.10) is relevant to describe percolation of gases through porous media [50], thin saturated regions in porous media, thin liquid films spreading under gravity [51], solid-on-solid model for surface growth [52], and superconductors [53]. For the particular case,

$$\lim_{t\to\infty}\Psi[x,t;\rho(x,t)] \to \text{const.} \quad \text{and} \quad \lim_{t\to\infty}F[x,t;\rho(x,t)] \to \text{const.}, \tag{2.15}$$

we find that equation (2.7) admits stationary solutions which, for general $\Psi[x;\rho_s(x)]$ and $F[x;\rho_s(x)]$, may be written in a self-consistent form as

$$\rho_s(x) = \frac{1}{\mathcal{Z}_s}\frac{1}{\Psi[x;\rho_s(x)]}\exp_q\left\{-\beta_s\int^x \mathrm{d}x'\,\frac{1}{\Psi^{1/\nu}[x';\rho_s(x')]}\frac{\partial}{\partial x'}V[x';\rho_s(x')]\right\}, \tag{2.16}$$

where $\nu = 2-q$ and $\beta_s = \mathcal{Z}_s^{\nu-1}/[(2-q)\mathcal{D}]$. In equation (2.16), $\mathcal{Z}_s$ and β_s are defined by the normalization of $\rho_s(x)$ and $F[x;\rho_s(x)] = -\partial_x V[x;\rho_s(x)]$, where $V[x;\rho_s(x)]$ satisfies the necessary conditions for the existence of a stationary solution.

If we use the Stratonovich interpretation, another nonlinear Fokker–Planck equation emerges from the Langevin equation (2.9), that is,

$$\frac{\partial}{\partial t}\rho(x,t)=\mathcal{D}\frac{\partial}{\partial x}\left\{\Psi[x,t;\rho(x,t)]\frac{\partial}{\partial x}\rho(x,t)\right\}-\frac{\partial}{\partial x}\left\{F[x,t;\rho(x,t)]\rho(x,t)\right\}, \tag{2.17}$$

which admits the stationary solution (under particular conditions)

$$\rho_s(x)=\frac{1}{\mathcal{Z}_s}\exp_q\left\{-\beta_s\int^x \mathrm{d}x'\,\frac{1}{\Psi[x';\rho_s(x')]}\frac{\partial}{\partial x'}V[x';\rho_s(x')]\right\}, \tag{2.18}$$

where $\exp_q[x]$ is the q-exponential function [28]:

$$\exp_q[x]\equiv\begin{cases}(1+(1-q)x)^{1/(1-q)}, & x>\dfrac{1}{(1-q)},\\ 0, & x<\dfrac{1}{(1-q)}.\end{cases} \tag{2.19}$$

The presence of the q-exponential function in the previous equation allows the identification of either a short- ($q<1$) or long-tailed ($q>1$) behaviour for the solution, depending on the value of q. Thus, equations (2.16) and (2.18) may have a compact behaviour for $q<1$ because of the cut-off imposed over the q-exponential to retain the probabilistic interpretation of the distribution. On the other hand, the solutions for $q>1$ may have asymptotic limits in the form of power-laws, and are thus related to Lévy distributions [54] and with solutions of fractional Fokker–Planck equations [49], which are also asymptotically power-laws.

Equation (2.17) is also directly related to the continuity equation

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}\mathcal{J}(x,t)=0, \tag{2.20}$$

if we assume that the current density is

$$\mathcal{J}(x,t)=-\mathcal{D}\Psi[x,t;\rho(x,t)]\frac{\partial}{\partial x}\rho(x,t)+F[x,t;\rho(x,t)]\rho(x,t). \tag{2.21}$$

Equation (2.21) indicates (on average) how fast the particles move to their preferential direction of motion. We can consider a similar situation for equation (2.7) by defining a suitable current density related to the particles' preferential direction of motion. In addition, reaction terms may be incorporated into equation (2.17) by modifying equation (2.20) to consider source or sink terms.

For the particular case where $\Psi[x,t;\rho(x,t)]=\nu(\delta/\eta)|x|^{1-\eta}t^{\delta-1}\rho(x,t)^{\nu-1}$, we can write equation (2.17) in the absence of external forces as

$$\frac{\partial}{\partial t}\rho(x,t)=\mathcal{D}\frac{\delta}{\eta}\frac{\partial}{\partial x}\left\{|x|^{1-\eta}t^{\delta-1}\frac{\partial}{\partial x}\rho(x,t)^{\nu}\right\}. \tag{2.22}$$

It is worth remarking that this equation presents fractal derivatives in the space and time variables. These derivatives were first developed by Chen [55] and Liang *et al.* [56] from the fractal concept and can be considered as an alternative approach to the fractional derivatives. We refer to ref. [57] for a discussion comparing the concepts of fractal and fractional derivatives. To make this relationship clearer, we observe that

$$\frac{1}{\delta t^{\delta-1}}\frac{\partial}{\partial t}\equiv\frac{\partial}{\partial t^{\delta}}\quad\text{and}\quad\frac{1}{\eta|x|^{\eta-1}}\frac{\partial}{\partial x}\equiv\frac{\partial}{\partial x^{\eta}}, \tag{2.23}$$

which after applying to equation (2.22) yield

$$\frac{\partial}{\partial t^{\delta}}\rho(x,t)=\mathcal{D}\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x^{\eta}}\rho(x,t)^{\nu}\right\}. \tag{2.24}$$

The solution for the former equation is

$$\rho(x,t)=\frac{1}{\Xi(t)}\exp_q\left[-\frac{k|x|^{\eta+1}}{\nu\mathcal{D}\Xi(t)}\right]\quad\text{with }\Xi(t)=\left[(1+\xi)kt^{\delta}\right]^{(1/(1+\xi))} \tag{2.25}$$

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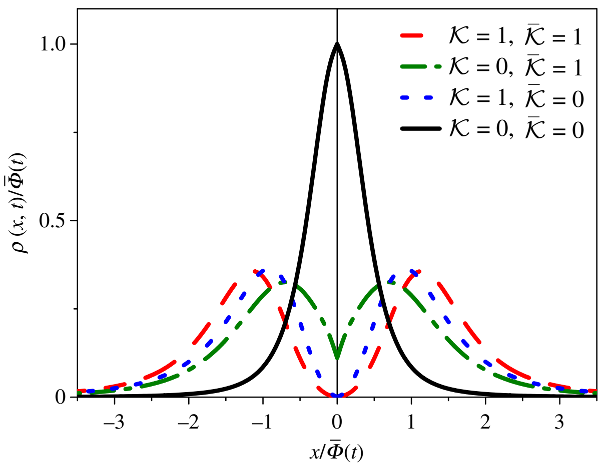

Figure 1. Behaviour of $\rho(x,t)/\bar{\Phi}(t)$ versus $x/\bar{\Phi}(t)$ for different values of $\mathcal{K}$ and $\bar{\mathcal{K}}$. For simplicity, we consider $\eta=1$ and the diffusion coefficients $D=1\ [L]^{1+\nu}/[T]$, $\nu=1/2$, where $[L]$ and $[T]$ represent arbitrary unities of length and time, respectively. (Online version in colour.)

where $\xi=\nu+\eta-1$, $\nu=2-q$, and k is defined by the normalization condition. The behaviour of the mean square displacement is $\langle(x-\langle x\rangle)^2\rangle\propto\Xi^2(t)$, i.e. $\langle(x-\langle x\rangle)^2\rangle\propto t^{(2/1+\xi)}$, leading to sub-, normal or superdiffusion depending on whether $2/(1+\xi)$ is less, equal or greater than one, respectively.

In equation (2.24), if we consider the external force

$$F(x,t)=-k(t)x+\left(\frac{\bar{\mathcal{K}}}{x}\right)|x|^{2-\eta-\nu}+\left(\frac{\mathcal{K}}{x}\right)|x|^{1-\eta}\rho^{\nu-1}(x,t), \tag{2.26}$$

where $k(t)$, $\bar{\mathcal{K}}$ and $\mathcal{K}$ are parameters controlling the intensity of the linear and nonlinear terms, the solution for $\rho(x,t)$ becomes

$$\rho(x,t)=\frac{1}{\bar{\Phi}(t)}\left|\frac{x}{\bar{\Phi}(t)}\right|^{(\mathcal{K}/\nu\mathcal{D})}\exp_q\left\{-\frac{k}{\nu\mathcal{D}\beta}\left|\frac{x}{\bar{\Phi}(t)}\right|^{\beta}+\frac{\bar{\mathcal{K}}}{\nu\mathcal{D}}\ln_{\bar{\nu}}\left|\frac{x}{\bar{\Phi}(t)}\right|\right\}, \tag{2.27}$$

where

$$\bar{\Phi}(t)=\bar{\Phi}(0)\left[1+\bar{k}\int_0^t \mathrm{d}t' t'^{\delta-1}\mathrm{e}^{(1-\xi)\int_0^{t'}\mathrm{d}\bar{t}k_1(\bar{t})\bar{t}^{\delta-1}}\mathrm{e}^{-(1-\xi)\int_0^{t}\mathrm{d}\bar{t}k_1(\bar{t})\bar{t}^{\delta-1}}\right]^{(1/(1+\xi))}, \tag{2.28}$$

$\beta=1+\eta+(\nu-1)\tilde{\mathcal{K}}/(\nu\mathcal{D})$, $\ln_\nu x=(x^{1-\nu}-1)/(1-\nu)$, and $\bar{\nu}=\nu+(\nu-1)\tilde{\mathcal{K}}/(\nu\mathcal{D})$ (see figure 1 for an illustration of behaviour of $\rho(x,t)/\bar{\Phi}(t)$). We notice that this system presents a stationary, i.e. a time-independent solution according to the function $k(t)$.

We can also include memory effects in our previous developments by considering a time dependence on Φ and modifying equation (2.6) as

$$\rho(x,t+\tau)=\int_0^t\int_{-\infty}^{\infty}\Psi[x-x',t';\rho(x-x',t')]\rho(x-x',t')\bar{\Phi}(x',t-t')\,\mathrm{d}x'\mathrm{d}t'$$
$$+\rho(x,t)-\int_0^t\Psi[x,t';\rho(x,t')]\rho(x,t')\mathcal{I}(t-t')\,\mathrm{d}t', \tag{2.29}$$

where $\mathcal{I}(t)=\int_{-\infty}^{\infty}\bar{\Phi}(x,t)\,\mathrm{d}x$. As the previous cases, we perform some expansions to obtain

$$\frac{\partial}{\partial t}\rho(x,t)=\int_0^t \mathrm{d}t'\bar{\mathcal{D}}(t-t')\frac{\partial^2}{\partial x^2}\left\{\Psi[x,t';\rho(x,t')]\rho(x,t')\right\}, \tag{2.30}$$

where $\bar{\mathcal{D}}(t)=[1/(2\tau)]\int_{-\infty}^{\infty}x'^2\tilde{\Phi}(x',t')\,\mathrm{d}x'$. By considering that $\tilde{\Phi}(x,t)=\Phi(x)\omega(t)$ with $\int_{-\infty}^{\infty}\mathrm{d}x\Phi(x)=1$, we find $\bar{\mathcal{D}}(t)=\mathcal{D}\mathcal{I}(t)$ (where $\mathcal{I}(t)=\omega(t)$) and equation (2.30) simplifies to

$$\frac{\partial}{\partial t}\rho(x,t)=\mathcal{D}\int_0^t \mathrm{d}t'\mathcal{I}(t-t')\frac{\partial^2}{\partial x^2}\left\{\Psi[x,t';\rho(x,t')]\rho(x,t')\right\}. \tag{2.31}$$

The time-dependent function $\mathcal{I}(t)$ determines the type of memory present in equation (2.31), besides the memory effects related to the nonlinear term. In the limit of $x\to\pm\infty$, an asymptotic solution for equation (2.31) with $\Psi[x,t;\rho(x,t)]=\rho^{\nu-1}(x,t)$ and $\mathcal{I}(t)=(t/\bar{\tau})^{\gamma}/[\Gamma(1+\gamma)\bar{\tau}]$ is $\rho(x,t)\sim(\bar{\Xi}(t)/|x|)^{\alpha'}/\bar{\Xi}(t)$, where $\alpha'=2/(1-\nu)$ and $\bar{\Xi}(t)\propto t^{\bar{\gamma}}$, and $\bar{\gamma}=(2+\gamma)/(1+\nu)$.

Within the framework of random walks, we may also consider two interacting walkers. As a prototype, we assume that one of the diffusing particles is immobilized by another particle that is in a resting position or even in motion. This system is represented within the random walk approach by two distributions: (i) $\rho_1(x,t)$ for one system of particles and (ii) $\rho_2(x,t)$ for the other system. To describe diffusive motion and to account for another general situation characterized by two walkers or two different species of particles, we replace equation (2.6) by two equations representing each distribution, that is,

$$\begin{aligned}\rho_1(x,t+\tau)=&\int_{-\infty}^{\infty}\mathrm{e}^{-k_1(t)\tau}\Psi_1[x-x',t;\rho_1(x-x',t)]\rho(x-x',t)\Phi_1(x')\,\mathrm{d}x'\\&+(1-\Psi_1[x,t;\rho_1(x,t)])\,\mathrm{e}^{-k_1(t)\tau}\rho_1(x,t)+\left(1-\mathrm{e}^{-k_2(t)\tau}\right)\mathcal{R}_{12}\left[\rho_2(x,t)\right]\\&+\left(1-\mathrm{e}^{-k_1(t)\tau}\right)\left(\rho_1(x,t)-\mathcal{R}_{11}\left[\rho_1(x,t)\right]\right)\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\rho_2(x,t+\tau)=&\int_{-\infty}^{\infty}\mathrm{e}^{-k_2(t)\tau}\Psi_2[x-x',t;\rho_2(x-x',t)]\rho(x-x',t)\Phi_2(x')\,\mathrm{d}x'\\&+(1-\Psi_2[x,t;\rho_2(x,t)])\,\mathrm{e}^{-k_2(t)\tau}\rho_2(x,t)+\left(1-\mathrm{e}^{-k_1(t)\tau}\right)\mathcal{R}_{21}\left[\rho_1(x,t)\right]\\&+\left(1-\mathrm{e}^{-k_2(t)\tau}\right)\left(\rho_2(x,t)-\mathcal{R}_{22}\left[\rho_2(x,t)\right]\right).\end{aligned} \tag{2.33}$$

The previous set of equations may be associated with the diffusion $\rightleftharpoons$ pauses process or a reaction diffusion process where $1\rightleftharpoons 2$ (with the rates $k_1(t)$ and $k_2(t)$). To verify this possibility, we consider the limits $\tau\to 0$ and $x'\to 0$, which yield

$$\frac{\partial}{\partial t}\rho_1(x,t)=D_1\frac{\partial^2}{\partial x^2}\left\{\Psi_1[x,t;\rho_1(x,t)]\rho_1(x,t)\right\}-k_1(t)\mathcal{R}_{11}\left[\rho_1(x,t)\right]+k_2(t)\mathcal{R}_{12}\left[\rho_2(x,t)\right] \tag{2.34}$$

and

$$\frac{\partial}{\partial t}\rho_2(x,t)=D_2\frac{\partial^2}{\partial x^2}\left\{\Psi_2[x,t;\rho_2(x,t)]\rho_2(x,t)\right\}+k_1(t)\mathcal{R}_{21}\left[\rho_1(x,t)\right]-k_2(t)\mathcal{R}_{22}\left[\rho_2(x,t)\right], \tag{2.35}$$

with $D_{1(2)}=\langle x'^2\rangle_{1(2)}/(2\tau)$. From the previous set of equations and by considering $\mathcal{R}_{11}[\rho_1(x,t)]=\mathcal{R}_{21}[\rho_1(x,t)]$ and $\mathcal{R}_{12}[\rho_2(x,t)]=\mathcal{R}_{22}[\rho_2(x,t)]$, we can show (under suitable boundary conditions) that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathrm{d}x\left\{\rho_1(x,t)+\rho_2(x,t)\right\}=0, \tag{2.36}$$

which implies in the conservation of the total number of walkers in the system. Equations (2.34) and (2.35) show a type of interaction between the walkers of system 1 and the walkers of system 2. Another way of interaction between systems 1 and 2 is through a diffusion that modifies the

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relaxation process and makes each system to act as a thermal bath to the other. This former situation can be modelled by considering

$$\Psi_1[x,t;\rho_1(x,t)] = \Lambda_2(t)\Upsilon_1[x;\rho_1(x,t)] \tag{2.37}$$

and

$$\Psi_2[x,t;\rho_1(x,t)] = \Lambda_1(t)\Upsilon_2[x;\rho_2(x,t)], \tag{2.38}$$

with $\Lambda_{1(2)}(t) = \int_{-\infty}^{\infty} \mathrm{d}x\mathcal{C}[\rho_{1(2)}(x,t)]$, where the function $\mathcal{C}_{1(2)}[\rho_{1(2)}(x,t)]$ determines how the interaction between systems 1 and 2 (see ref. [42] for a particular case) is. By using equations (2.37) and (2.38), we can write the nonlinear equations as

$$\frac{\partial}{\partial t}\rho_1(x,t) = D_1\Lambda_2(t)\frac{\partial^2}{\partial x^2}\{\Upsilon_1[x;\rho_1(x,t)]\rho_1(x,t)\} - k_1(t)\mathcal{R}_{11}[\rho_1(x,t)] + k_2(t)\mathcal{R}_{12}[\rho_2(x,t)] \tag{2.39}$$

and

$$\frac{\partial}{\partial t}\rho_2(x,t) = D_2\Lambda_1(t)\frac{\partial^2}{\partial x^2}\{\Upsilon_2[x;\rho_2(x,t)]\rho_2(x,t)\} + k_1(t)\mathcal{R}_{21}[\rho_1(x,t)] - k_2(t)\mathcal{R}_{22}[\rho_2(x,t)]. \tag{2.40}$$

To go further on, we consider the illustrative case where $\mathcal{R}_{22}[\rho_2(x,t)] = \mathcal{R}_{21}[\rho_2(x,t)] = 0$, $\mathcal{R}_{11}[\rho_1(x,t)] = \mathcal{R}_{12}[\rho_1(x,t)] = \rho_1(x,t)$, $\Upsilon_1[x;\rho_1(x,t)] = |x|^{-\eta_1}\rho_1^{\nu_1-1}(x,t)$, $\Upsilon_2[x;\rho_2(x,t)] = \rho_2^{\nu_2-1}(x,t)$ and $\mathcal{C}_{1(2)}[\rho_{1(2)}(x,t)] = \rho_{1(2)}^{\nu_{1(2)}}(x,t)$, with $\nu_1 \neq 1$ and $\nu_2 = 1$. Thus, the two previous equations become, respectively,

$$\frac{\partial}{\partial t}\rho_1(x,t) = D_1\Lambda_2(t)\frac{\partial^2}{\partial x^2}\left\{|x|^{-\eta_1}\rho_1^{\nu_1}(x,t)\right\} - k_1(t)\rho_1(x,t) \tag{2.41}$$

and

$$\frac{\partial}{\partial t}\rho_2(x,t) = D_2\Lambda_1(t)\frac{\partial^2}{\partial x^2}\rho_2(x,t) + k_1(t)\rho_1(x,t). \tag{2.42}$$

To obtain the solution for this set of equations, we first use the method of similarity to tackle equation (2.41), i.e. the solutions are scaled functions as

$$\rho_1(x,t) = \mathrm{e}^{-\int_0^t \mathrm{d}t' k_1(t')}\frac{\bar{\rho}_1(z)}{\Xi_1(t)}, \tag{2.43}$$

where $z = x/\Xi_1(t)$. The substitution of equation (2.43) into equation (2.41) yields the transformed equations

$$\Xi_1^{\nu_1+\eta_1}(t)\frac{\mathrm{d}}{\mathrm{d}t}\Xi_1(t) = kD_1\Lambda_2(t)\mathrm{e}^{(1-\nu_1)\int_0^t \mathrm{d}t' k_1(t')} \tag{2.44}$$

and

$$-k\frac{\mathrm{d}}{\mathrm{d}z}[z\bar{\rho}_1(z)] = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{|z|^{-\eta_1}\bar{\rho}_1(z)\right\}, \tag{2.45}$$

whose solutions are, respectively,

$$\Xi_1(t) = \Xi(0)\left\{1 + \bar{k}\int_0^t \mathrm{d}t'\Lambda_2(t')\mathrm{e}^{(1-\nu_1)\int_0^{t'} \mathrm{d}t'' k_1(t'')}\right\}^{(1/(1+\eta_1+\nu_1))} \tag{2.46}$$

and

$$\bar{\rho}_1(z) = |z|^{\eta_1/\nu_1}\exp_{q_1}\left[-\frac{k|z|^{2+\eta_1/\nu_1}}{(2\nu_1+\eta_1)D_1}\right], \tag{2.47}$$

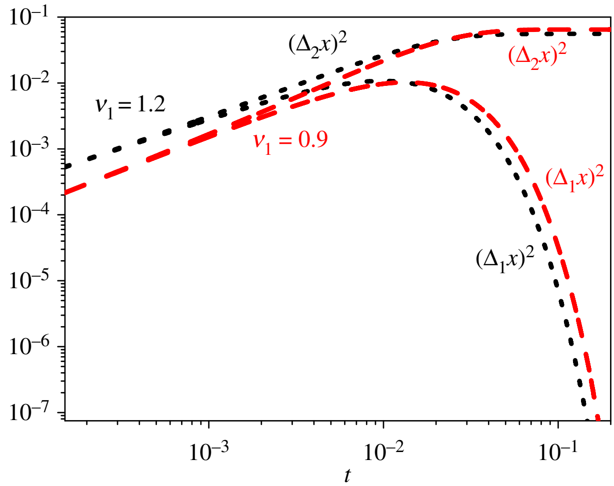

Figure 2. Behaviour of the mean square displacement $(\Delta_{1(2)}x)^2 = \langle (x - \langle x \rangle)^2 \rangle_{1(2)}$ for the distributions $\rho_1(x,t)$ and $\rho_2(x,t)$. These curves were drawn for $\eta_1 = 0$, with constant diffusion coefficients $D_1 = 1\ [L]^{1+\nu_1}/[T]$ and $D_2 = 1\ [L]^2/[T]$, and constant rate $k_1(t) = 1\ [T]^{-1}$, where $[L]$ and $[T]$ represent arbitrary unities of length and time. (Online version in colour.)

where $\bar{k} = (1 + \eta_1 + \nu_1)k/\Xi_1^{\eta_1+\nu_1}(0)$ and $q_1 = 2 - \nu_1$. We further note that $\Lambda_2(t) = \Lambda_2(0) + \mathcal{I}_1(0)[1 - \mathrm{e}^{-\int_0^t \mathrm{d}t' k_1(t')}]$, where $\mathcal{I}_1(t) = \mathcal{I}_1(0)\mathrm{e}^{-\int_0^t \mathrm{d}t' k_1(t')}$ and $\mathcal{I}_1(0) = \int_{-\infty}^{\infty} \mathrm{d}x \rho_1(x,0)$.

For the solution of equation (2.42), we have

$$\rho_2(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x' \rho_2(x',0)\mathcal{G}_2(x - x', \mathcal{T}_2(t)) + \int_0^t \mathrm{d}t' k_1(t') \int_{-\infty}^{\infty} \rho_1(x',t')\mathcal{G}_2(x - x', \mathcal{T}_2(t) - \mathcal{T}_2(t')), \tag{2.48}$$

where $\mathcal{T}_2(t) = \int_0^t \mathrm{d}t' \Lambda_1(t')$ and

$$\mathcal{G}_2(x, \mathcal{T}_2(t)) = \frac{1}{\sqrt{4\pi D_2 \mathcal{T}_2(t)}} \exp\left(-\frac{x^2}{4D_2\mathcal{T}_2(t)}\right). \tag{2.49}$$

From the previous solutions, we calculate mean square displacements related to equations (2.43) and (2.48), as illustrated in figure 2. This figure shows that different values of ν_1 initially produce different relaxations for $\rho_2(x,t)$ with the same stationary state.

Finally, we consider another relevant situation obtained from equations (2.34) and (2.35) with only a sink term, i.e. $\mathcal{R}_{22}[\rho_2(x,t)] = \mathcal{R}_{21}[\rho_2(x,t)] = \mathcal{R}_{12}[\rho_1(x,t)] = 0$, $\mathcal{R}_{11}[\rho_1(x,t)] = \rho_1(x,t)$ for $\rho_1(x,t)$,

$$\Psi_1[x,t;\rho_1(x,t)] = \Lambda_2(t)\rho_1^{\nu_1-1}(x,t) \quad \text{and} \quad \Psi_2[x,t;\rho_2(x,t)] = (1 + \alpha\Lambda_1(t))\rho_2^{\nu_2-1}(x,t), \tag{2.50}$$

with $\nu_1 \neq 1$ and $\nu_2 = 1$. In this case, the walkers in system 1 may be considered as an additional thermal bath on walkers 2, which are removed at the rate $k_1(t)$. The solutions for this case are obtained by following a similar approach and the mean square displacement for $\rho_2(x,t)$ is illustrated in figure 3. The spreading of the walkers in system 2 presents different diffusive regimes before reaching the usual one characterized by a linear time dependence. In particular, the subdiffusive or superdiffusive regimes for $\rho_2(x,t)$ are obtained for small times depending on the values of ν_1.

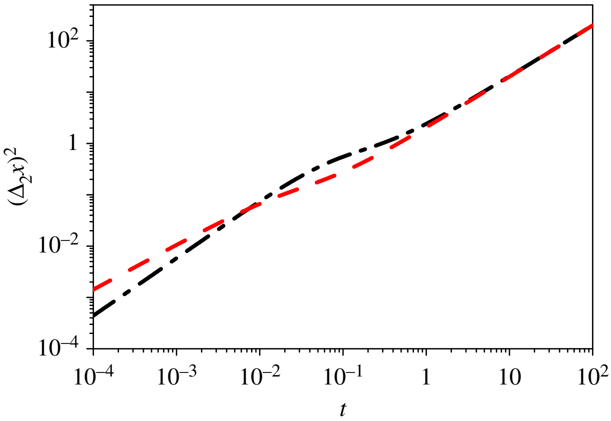

Figure 3. Behaviour of the mean square displacement, $(\Delta_2 x)^2 = \langle (x - \langle x \rangle)^2 \rangle_2$, for the distribution $\rho_2(x,t)$ to highlight the influence of the interaction among the particles. The black line represents the case $\eta_1 = 1$ with $\nu_1 = 1/2$. The red line represents the case $\eta_1 = 1.0$ with $\nu_1 = 3/2$. For simplicity, we consider $\eta_1 = 1$, diffusion coefficients $D_1 = 1\ [L]^{1+\nu_1+\eta_1}/[T]$ and $D_2 = 1\ [L]^2/[T]$, $\alpha = 2$, and the rate $k_1(t) = 50\ [T]^{-1}$, where $[L]$ and $[T]$ represent arbitrary unities of length and time. (Online version in colour.)

3. Discussion and conclusion

The analysis presented here has focused on how to use the random walk approach to obtain nonlinear diffusion equations. Our discussion has started by revisiting the derivation of Einstein for the usual diffusion equation, and next by modifying this procedure for obtaining different nonlinear diffusion equations. We have discussed how this procedure is connected with Ito–Langevin equations and the changes obtained with the Stratonovich approach for the stochastic noise. We have also associated the spatial dependence of the diffusion coefficient with the fractal derivatives, which in turn highlights the connection between the spatial heterogeneity and fractals. In this case, the solution (equation (2.25)) was obtained by using the method of similarity, and the mean square displacement showed anomalous spreading with sub-, super- and normal diffusive regimes. We have also observed that the solution of this problem is asymptotically related to the Lévy distributions. Similarly, we have also considered the external force given equation (2.26) in equation (2.17). In this case, we have found the exact solution (see equation (2.25)) showed that time-independent (stationary) solution exists for $t \to \infty$.

We have further accounted for the interaction among the walkers in a quite natural way for two representative situations. In the first one, we have considered reaction terms capable of converting one kind of walker into the other at a given rate. We have also discussed how this process can describe a situation where one type of walker is immobilized during the diffusive process. The results we present here have as particular cases those presented in ref. [42]. In the second situation, we have dealt with the presence of only a sink term in a way that one of the walkers (e.g. walker 1) represents a thermal bath for the other walkers (e.g. walker 2), which are removed at a time-dependent ratio.

Data accessibility. This article has no additional data.

Authors' contributions. E.K.L., M.K.L., H.V.R. and L.R.E. wrote the paper. E.K.L. and L.R.E. performed the calculation and M.K.L. and H.V.R. verified the calculations. E.K.L. made the figures. M.K.L. and H.V.R. revised the paper.

Competing interests. We declare we have no competing interest.

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